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Fluctuations in Nonequilibrium Systems and Broken Supersymmetry

Michael F. Zimmer[1]

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The fluctuation-dissipation theorem is not expected to hold for systems that either violate detailed balance or have time-dependent or nonpotential forces. Therefore the relation between response and correlation functions should have contributions due to the nonequilibrium nature. An explicit formula for such a contribution is calculated, which in the present derivation appears as a history-dependent term. These relations are the Ward–Takahashi identities of a supersymmetric formulation of the Langevin models, and the new term results from a broken supersymmetry.

KEY WORDS: Fluctuation-dissipation theorem; nonequilibrium systems; supersymmetry; Ward–Takahashii identity; detailed balance; driven diffusive system; nonpotential forces; BRS symmetry; mode coupling.

1. INTRODUCTION

While there is already a large amount of information about systems near equilibrium, correspondingly little is known about those far from equilibrium. In particular, for equilibrium systems we know the distribution function for fluctuations, as well as identities relating the correlation of fluctuations to dissipation, known as the fluctuation-dissipation theorem (FDT) of the first kind.[1,2] In nonequilibrium systems, neither the distribution function nor any explicit (and general) relations amongst correlations have been found.

In near-equilibrium systems modeled by Langevin equations, it can be shown that the property of time-reversal invariance (TRI) leads to detailed balance (DB), from which the FDT follows.[3,4] Thus in nonequilibrium

[1] Department of Physics, Materials Research Laboratory, and Beckman Institute University of Illinois at Urbana-Champaign, Illinois, U.S.A. E-mail:mike@kolmogorov.physics.uiuc.edu.

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systems that violate TRI or DB, it is expected that the FDT will be violated. To the author's knowledge, the only times such identities have been found for systems violating DB were in relatively simple special cases.[5,6] There has also been a rather abstract but general work discussing the existence of the FDT for nonequilibrium systems,[7] but it seems that no explicit identities have resulted from it. Finally, in another work[5] the recognition that SUSY must be broken for a system out of equilibrium was made. However, they did not fully exploit this realization to derive the identities found here.

In this paper I calculate explicit identities relating the correlation and response functions for several nonequilibrium models. Working from a field-theoretic formulation and utilizing a broken supersymmetry (SUSY), I find Ward–Takahashi identities (WTIs)[8] that describe these systems. They consist of the usual FDT plus a new contribution due to terms which keep the system from equilibrium. These terms are not incorporated into the usual equilibrium distribution and seem to participate as additional sources in the system, acting on the same level as thermal fluctuations.

Three fairly general nonequilibrium models are considered here, and the above-mentioned WTIs are calculated for each. The first example is a spin system forced from equilibrium by a time-dependent external field.[9–11] Such a system specifically models experiments on phase transitions[12] and even industrial machinery[2]; is also acts as a paradigm for general systems under time-dependent fields. The second example is a generic Langevin model with forces violating detailed balance. This case has many examples, including Ising spin glasses,[13] neural nets with asymmetric bonds,[14] and fluids under shear flow.[15] Finally, I derive a generalized FDT for a driven diffusive system (see ref. 16 for review), i.e., a conserved order parameter under the influence of an external electric field; it is meant to model fast ionic conductors (see, e.g., ref. 17). The applicability of this approach for mode-coupling[18] models is also discssed.

The existence of a hidden SUSY ion Langevin equations describing equilibrium systems has been known for some time.[19] SUSY is actually composed of two independent symmetries, a BRS[20] symmetry and a second, fermionic symmetry (see explanation in ref. 21). In quite general terms, the generating functional for correlations of fields (denoted collectively by Ψ, with source $\mathscr{T}$) is

$$Z[\mathscr{T}] = \int \mathscr{D}\Psi \exp\left[-S(\Psi) + \int dx\, \mathscr{T} \cdot \Psi\right] \tag{1}$$

[2] Machinery that is very similar to this system, which is used to thermally anneal metallic parts, is produced by Magnatech, Inc., 5790 Fenno Road, Bettendorf, Iowa 55272.

where S is the action. An infinitesimal linear variation of the fields [i.e., $\delta\Psi(\Psi)$] causes the action to vary by $\delta S(\Psi)$, and quite directly leads to[3]

$$\left\langle -\delta S + \int dx\, \mathscr{T} \cdot \delta\Psi \right\rangle_{\mathscr{T}} = 0 \tag{2}$$

Here $\langle \cdot \rangle_{\mathscr{T}}$ denotes a noise average with sources $\mathscr{T}$ present ($\langle \cdot \rangle$ is defined as $\langle \cdot \rangle_{\mathscr{T}=0}$). Systems that are invariant under this linear transformation have $\delta S = 0$, and are said to possess one or the other symmetry. In the examples considered here, I find that $\delta S = 0$ for the BRS symmetry, which is expected since this symmetry is associated with causality. However, for the second fermionic symmetry, $\langle \delta S \rangle_{\mathscr{T}} \neq 0$, for each example. This is also sensible, since this symmetry seems to be linked with detailed balance, although the exact relationship has never been elucidated. However, it is known that potentiality conditions are needed to be able to write the action in supersymmetric form. In any case, from Eq. (2) identities between correlation functions can be bound in the usual way. One differentiates the equation with respect to the appropriate sources (at different points), and then sets all sources to zero.

2. OSCILLATING MAGNETIC FIELD

In this case I consider the fluctuations of a spin system forced from equilibrium by the time-dependent external field $h(t)$. For this particular case the external field is coupled linearly to the spin order parameter field $\psi(\mathbf{r}, t)$. The system is modeled by the time-dependent Ginzburg–Landau equation (TDGL)[18]

$$\partial_t \psi = -\Gamma_0 \frac{\delta H}{\delta \psi} + v \tag{3}$$

$$H = \int d^d r \left\{ \frac{1}{2} \left[r_0 \psi^2 + (\nabla\psi)^2 \right] + \frac{u_0}{4} \psi^4 - h(t)\psi \right\} \tag{4}$$

with zero-mean equilibrium noise correlations given by $\langle v(x)\, v(x') \rangle = 2\Gamma_0 \delta(x - x')$, $x \equiv (\mathbf{r}, t)$. In the following I will focus on the special case $h(t) = h \cos \Omega t$.

[3] For general linear transformations it is possible to have a field-independent contribution from the Jacobian of transformation. However, in most cases it will not contribute to a WTI, since the equation usually first has to be differentiated with respect to sources. But in any case, this is not an issue for the BRS and fermionic symmetries, since the Jacobian does not contribute to Eq. (2).

Once the system relaxes from its initial conditions, it settles into a periodic state $M(t)$, with period $2\pi/\Omega$. A steady state may be projected from this by either considering timeslices every $2\pi/\Omega$, or by putting the theory on a lattice, treating $M(t)$ integrated over one period as the field variable. However, since it is the fluctuations about $M(t)$ which are of primary interest, I shift $\psi \to \psi + M(t)$ and define $M(t)$ through the condition $\langle \psi \rangle = 0$. The model now reads

$$\partial_t \psi - \Gamma_0 \frac{\delta A}{\delta \psi} - F[M] + v \tag{5}$$

$$\frac{\delta A}{\delta \psi} = (r_0 - \nabla^2)\psi + u_0(\psi^3 + 3\psi^2 M + 3\psi M^2) \tag{6}$$

$$F[M] = \partial_t M + \Gamma_0[r_0 M + u_0 M^3 - h(t)] \tag{7}$$

ψ now measures deviations about $M(t)$. In this nonlinear case there remain time-dependent terms in the restoring force $\delta A/\delta\psi$. These will govern relaxations of thermal fluctuations, and thus it is seen there is a lack of TRI in the system. Because TRI implies DB, which in turn implies the FDT of the first kind, a violation of the FDT is expected. It is for this reason that the following field-theoretic approach to calculating amendments to the FDT is considered.

The convenience of introducing auxiliary fields as bookkeeping devices ie a perturbation expansion has long been realized. Using such an approach on this system, one can write the generating functional for the spin field ψv the response (or MSR[22,23]) field $\tilde{\psi}$, and two anticommuting (Grassmann[24]) fields C and $\bar{C}$ [defining $C_x = C(\mathbf{r}, t)$]:

$$Z[J, \tilde{J}, K, \bar{K}] = \int \mathscr{D}\psi\, \mathscr{D}(i\tilde{\psi})\, \mathscr{D}C\, \mathscr{D}\bar{C} \times \exp\left(-S + \int |J\psi + \tilde{J}\tilde{\psi} + KC + \overline{KC}] \right) \tag{8}$$

$$S = \int dx \left\{ \tilde{\psi} \left(\dot{\psi} + \Gamma_0 \frac{\delta A}{\delta \psi} + F \right) - \Gamma_0 \tilde{\psi}^2 - \bar{C} \int dx' \left[\partial_t \delta(x - x') + \Gamma_0 \frac{\delta^2 A}{\delta \psi_{x'}\, \delta \psi} \right] C_{x'} \right\} \tag{9}$$

In addition, appropriate source terms are introduced for each field. As is well known, a symmetry in the action can often be used to derive nonperturbative identities. For Langevin equations, it is known there are two such symmetries, one of which only becomes obvious when the action is written in superspace. Proceeding in that direction with the superfield

$$\phi = \psi + \bar{\theta} C + \bar{C} \theta + \bar{\theta} \theta \tilde{\psi} \tag{10}$$

where $\bar{\theta}, \theta$ are anticommuting (Grassmann) coordinates,we find that the action becomes

$$S = \int d\bar{\theta}\, d\theta \left\{ \int dx \left[\frac{\partial \phi}{\partial \theta} \left(\theta \frac{\partial \phi}{\partial t} - \Gamma_0 \frac{\partial \phi}{\partial \bar{\theta}} \right) - \phi F \right] - \Gamma_0 \int dt\, A(\phi, t) \right\} \tag{11}$$

For equilibrium theories the action possesses a translational invariance in $\bar{\theta}$ as well as a sort of Galilean invariance involving $\bar{\theta}$, θ, and t. The generators for these symmetries are $Q = \partial_{\bar{\theta}}$ and $Q' = \partial_\theta + (\bar{\theta}/\Gamma_0)\partial_t$, respectively, which together generate the supersymmetry. That is, if the fields are varied by $\delta\phi = (\varepsilon Q + Q'\varepsilon')\phi$ and then compensated with the coordinate transformation

$$(\theta, \bar{\theta}, t) \rightarrow \left(\theta + \varepsilon', \bar{\theta} - \varepsilon, t + \varepsilon' \frac{\bar{\theta}}{\Gamma_0} \right) \tag{12}$$

the form of the action is recovered. It is easily shown that the Jacobian of this transformation is one.

As mentioned earlier, these symmetries may be used to derive WTIs. The symmetry with generator Q (a BRS symmetry) yields the equation[25]

$$\int dx\, \langle JC - \bar{K}\tilde{\psi} \rangle_{\mathcal{J}} = 0 \tag{13}$$

By differentiating with respect to J_2 and $\bar{K}_1$ [where $J_2 = J(\mathbf{r}_2, t_2)$, etc.], and then setting the sources to zero, it follows that $\langle C_1 \bar{C}_2 \rangle = -\langle \psi_1 \tilde{\psi}_2 \rangle$, as well as other identities such as $\langle \tilde{\psi} C \rangle = \langle \tilde{\psi} \bar{C} \rangle = \{ \psi C \rangle = \langle C \rangle = \langle \tilde{\psi} \rangle = 0$. The first identity ensures that closed loops of propagators $\propto \theta(0)$ [$\theta(x)$ is the Heaviside function] cancel in a perturbation expansion. When ϕ is varied with Q' and then is followed by a compensating coordinate transformation, the action changes by an amount (due to the coordinate transformation)

$$\delta S = -\varepsilon' \int dx\, \bar{C} \left[\frac{1}{\Gamma_0} \dot{F} + \mathscr{F}(\psi) \right], \qquad \mathscr{F}(\psi) = \dot{M} \frac{\partial}{\partial M} \frac{\delta A}{\delta \psi} \tag{14}$$

which by definition breaks SUSY (the overdot denotes time differentiation). The form of this term makes it clear that it is the explicit time dependence in $M(t)$ which is responsible for this symmetry breaking. Similar to before, the following equation may be derived:

$$\left\langle \frac{1}{\varepsilon'} \delta S \right\rangle_{\mathscr{J}} = \int dx \left\langle J\bar{C} + \frac{1}{\Gamma_0} \tilde{J}\dot{\bar{C}} + K\left(\tilde{\psi} - \frac{1}{\Gamma_0} \dot{\psi}\right)\right\rangle_{\mathscr{J}} \tag{15}$$

Differentiating this with respect to J_2 and K_1 and then setting the sources to zero, I find the following equation

$$\frac{1}{\Gamma_0} \partial_{t_1} \langle \psi_1 \psi_2 \rangle - \langle \psi_2 \tilde{\psi}_1 \rangle + \langle \psi_1 \tilde{\psi}_2 \rangle = \left\langle \psi_2 C_1 \int dx\, \bar{C} \left(\frac{1}{\Gamma_0} \dot{F} + \mathscr{F}(\psi) \right) \right\rangle \tag{16}$$

which contains the equilibrium FDT on the LHS,[4] plus a new contribution on the RHS. In this model $\mathscr{F}(\psi) = 3u_0[\dot{M}\psi^2 + 2M\dot{M}\psi]$.

The lowest order contribution to the RHS of Eq. (16) comes from the only possible Wick factorization. Using $\langle \psi \rangle = 0$, the RHS becomes, at *lowest* order, $-\int dx\, \langle \psi_1 \tilde{\psi}_x \rangle \langle \mathscr{F}(\psi_x) \psi_2 \rangle$. Evaluated to $\mathscr{O}(u_0^2)$, this equals

$$-3u_0 C_0(t_1, t_2)\, M^2(t_1) + 6u_0 \Gamma_0 \int dt'\, G_0(t_1, t')\, G_0(t_2, t')\, M^2(t') + \mathscr{O}(u_0^2) \tag{17}$$

and agrees with a calculation of the LHS of Eq. (16) to the same order. Here $G_0(t_2, t_1)$ is the response function $\theta(t_2 - t_1) e^{-\omega(t_2 - t_1)}$, $C_0(t_1, t_2)$ is the correlation function $(\Gamma_0/\omega\omega)[G_0(t_1, t_2) + G_0(t_2, t_1)]$, and $\omega = r_0 \Gamma_0$. As required, the above expression vanishes if $M(t)$ is constant. Also, it is not so useful to Fourier transform this, since it is no longer true that $\langle \psi_{\omega_1} \psi_{\omega_2} \rangle \propto \delta(\omega_1 + \omega_2)$, due to the time dependence in $\delta A / \delta \psi$.

A cursory analysis of thihs equation shows that the spontaneous fluctuations are still limited by the "pliability" of the system, which can be measured by finding the response to an external disturbance. Then the new equation merely expresses the complication of broken time-translational invariance. In our models there is a broken DB, which, when unbroken, is the statement that the flux of probability between any two states vanishes. This is a sufficient condition to ensure that the rate of change of the population P of a given state n vanishes, i.e.,

$$\partial_t P_n = \sum_{n'} (J_{nn'} - J_{n'n}) = 0 \tag{18}$$

where J_{ab} is the flux of probability from state b to a. Loosely speaking, one can say that the fluctuations cancel in such a way that the equilibrium

[4] In equilibrium the FDT has been derived via SUSY in refs. 26.

distribution is recovered. Hence in a measurement of $\langle \psi_2 \dot{\psi}_1 \rangle\, \theta(t_2 - t_1)$ [or essentially $\langle \psi_2 v_1 \rangle$], there should not be a contribution from times before t_1. It is easy to see that this is guaranteed by the usual FDT

$$\theta(t_2 - t_1)\langle \psi_2 \dot{\psi}_1 \rangle \sim \langle \psi_2 \tilde{\psi}_1 \rangle \tag{19}$$

whose RHS is essentially built of propagators that can only evolve forward in time. However, in a system that violates DB, there is a residual flux (i.e., $J_{nn'} - J_{n'n} \neq 0$) which, although it may cancel after summing over n', is still present in the system. Based on our equation, it seems that perhaps the term can be thought of as source in the system, stirring it up and keeping it out of equilibrium. Since this source is presumably present at all times, it has the potential to influence spontaneous fluctuations at later times. This is an explanation for why the additional term in Eq. (16) is integrated over the history of the system. Finally, it is possible to derive other identities from Eq. (15) that show more explicitly how the nonequilibrium terms "stir" the system. Differentiating the equation with respect to K_1, we get

$$\langle \dot{\psi}_1 \rangle = \Gamma_0 \left\langle C_1 \int dx\, \bar{C}[\dot{F}/\Gamma_0 + \mathscr{F}(\psi)] \right\rangle \tag{20}$$

which of course would vanish in an equilibrium theory.

Before proceeding on to the following subsection, the general approach for finding WTIs is recapitulated, as it will be of use for the next two examples. For models of the form

$$\partial_t \varphi = -\Gamma F(\varphi) + v \tag{21}$$

$$\langle v_x v_{x'} \rangle = 2\Gamma \delta(x - x') \tag{22}$$

it is possible to derive an MSR action and then rewrite it in superspace as

$$S = \int d\bar{\theta}\, d\theta \int dx \left\{ D\phi D'\phi - \Gamma\theta \frac{\partial \phi}{\partial \theta} F(\phi) \right\} \tag{23}$$

where ϕ is the usual superfield and $D = \partial_\theta$, $D' = (\theta\partial_t - \Gamma\partial_{\bar{\theta}})$. In the special case where $F(\phi)$ can be written as $\delta H/\delta\phi$, the action takes the form

$$S = \int dt \int d\bar{\theta}\, d\theta \left\{ \int d^d r\, D\phi D'\phi - \Gamma H(\phi) \right\} \tag{24}$$

This form is clearly supersymmetric; it is possible to vary the coordinates according to Eq. (12) (under which D, D' are invariant) and then vary the fields to cancel the induced field transformation. If there is an explicit time dependence in F, then a contribution due to the coordinate transformation will not cancel against the field variation, thus breaking SUSY (or really just the second fermionic symmetry). Likewise, if Eq. (23) cannot be rewritten into the form of Eq. (24), then the lone factor of θ ruins the second symmetry, and so again SUSY is broken; this is known to occur for systemms lacking DB, for example. But in either case the BRS symmetry is maintained, which is what one would expect, since causality is still present in these nonequilibrium models. Finally, since it is now apparent how extra terms come about from a broken SUSY, it will no longer be necessary to rewrite the action in terms of superfields; instead the four independent fields may be varied appropriately. This is the approach that will be taken on the remaining examples.

Power Dissipation

Quite generally, when a force $f(t)$ is applied to a system that is otherwise at equilibrium, a power (per unit volume) $f(t)\,\partial_t R_f(t)$ is expended, where $R_f(t) = \int^t dt'\,\chi(t, t')\,f(t')$ and χ is the susceptibility; this holds independent of any linearity assumptions on χ. However, in the case when there is already an external field $h(t)$ present, the power that is dissipated due to $f(t)$ equals

$$p(t) = (h+f)\,\partial_t R_{h+f} - h\partial_t R_h \tag{25}$$

$$= (h+f)\,\partial_t[R_{h+f} - R_h] + f\partial_t R_h \tag{26}$$

It can be shown (at a perturbative level) in this model that $R_{h+f} - R_h$ is the response to $f(t)$ *about* the periodic state M-(t). Now, the susceptibility about the equilibrium state can be measured and gives R_h. When $(h+f)$ is being applied, $p(t)$ can be measured and thus the susceptibility about the periodic state can be measured. This can be used, in conjunction with a measurement of $\langle \dot{\psi}_1 \psi_2 \rangle$ (via inelastic scattering), to measure the new term that amends the usual FDT.

In the equilibrium case the power dissipated can be related to the imaginary part of the susceptibility, and so the name of the fluctuation-dissipation theorem is justified. Here such a simple relation does not exist for a system forced from equilibrium by a time-dependent external field. As already mentioned, in nonequilibrium systems a spontaneous fluctuation is compresed not only of a response to a thermal fluctuation, but also a history-dependent term that is related to a violation of DB.

3. NONPOTENTIAL FORCES

As mentioned earlier, there are Langevin models used throughout physics that violate DB. Here I consider a rather general form of a model with a part ($\propto L_{\alpha,\beta} = L_{\beta\alpha}$)[5] which satisfies DB and a part (N_α) which in general does not; in particular, N_α cannot be derived from a potential. The model reads

$$\partial_t \psi_\alpha = -F_\alpha + v_\alpha \tag{27}$$

$$-F_\alpha = N_\alpha(\psi) - L_{\alpha\beta} \frac{\delta H}{\delta \psi_\beta} \tag{28}$$

$$\langle v_\alpha(x)\, v_\beta(x') \rangle = 2L_{\alpha\beta} \delta(x - x') \tag{29}$$

As done earlier, this problem may be formulated field-theoretically, by introducing auxialiary fields. In this case the action is

$$S = \int dx \left\{ \tilde{\psi}_\alpha(\dot{\psi}_\alpha + F_\alpha) - \tilde{\psi}_\alpha L_{\alpha\beta} \tilde{\psi}_\beta - \bar{C}_\alpha \int dy \left[\partial_t \delta_{\alpha\gamma}(x - y) + \frac{\delta F_\alpha}{\delta \psi_\gamma} \right] C_\gamma \right\} \tag{30}$$

Before, this would have been written in superspace, and then field and coordinate transformations would have been made to elicit WTIs.[6] But since it is now known what the appropriate superfield transformation is [i.e., $\delta\phi = (\varepsilon Q + Q'\varepsilon')\phi$], the component form of $\delta\phi$,

$$\begin{aligned} \delta\psi_\alpha &= \varepsilon C_\alpha + \varepsilon' L_{\beta\alpha} \bar{C}_\beta, \qquad \delta C_\alpha = \varepsilon'(\dot{\psi}_\alpha - L_{\beta\alpha} \tilde{\psi}_\beta) \\ \delta\tilde{\psi}_\alpha &= \varepsilon' \dot{\bar{C}}_\alpha, \qquad \delta \bar{C}_\alpha = \varepsilon \tilde{\psi}_\alpha \end{aligned} \tag{31}$$

may instead be used directly in a variation of S. Doing this leads to two equations which can be used to generate WTIs. From the ε variation follows

$$\int dx\, \langle J_\alpha C_\alpha - \bar{K}_\alpha \tilde{\psi}_\alpha \rangle_{\mathscr{J}} = 0 \tag{32}$$

[5] The physical importance of this choice was first discussed by Onsager.[27]

[6] By defining $\phi'_\alpha = \psi_\alpha + \bar{\theta} C_\alpha - \bar{C}'_\alpha \theta + \bar{\theta}\theta\tilde{\psi}'_\alpha$, where $\tilde{\psi}'_\alpha = L_{\beta\alpha}\tilde{\psi}_\beta$ and $\bar{C}'_\alpha = L_{\beta\alpha}\bar{C}_\beta$, it is possible to write the action as

$$S = S^{\mathrm{SUSY}} + S'$$

$$S^{\mathrm{SUSY}} = \int dt \int d\bar{\theta}\, d\theta \left\{ L_{\beta\alpha}^{-1} \int d^d r \frac{\partial \phi'_\beta}{\partial \theta} \left(\theta \frac{\partial \phi'_\alpha}{\partial t} - \frac{\partial \phi'_\alpha}{\partial \bar{\theta}} \right) - H(\phi') \right\}$$

$$S' = \int dx \int d\bar{\theta}\, d\theta \left\{ L_{\beta\alpha}^{-1} \theta \frac{\partial \phi'_\beta}{\partial \theta} N_\alpha(\phi') \right\}$$

From this it is easily seen that will be an extra term from the shift in θ. This will given the same contribution to $\langle \delta S \rangle_{\mathscr{J}}$ as found in Eq. (35).

and from the ε' variation

$$\frac{1}{\varepsilon'}\langle\delta S\rangle_{\mathscr{I}}=\int dx\,\langle J_\alpha L_{\beta\alpha}\bar{C}_\beta+\tilde{J}_\alpha\dot{\bar{C}}_\alpha K_\alpha(L_{\beta\alpha}\tilde{\psi}_\beta-\dot{\psi}_\alpha)\rangle_{\mathscr{I}} \tag{33}$$

where (omitting integration symbols)

$$\frac{1}{\varepsilon'}\delta S=\tilde{\psi}_\alpha\frac{\delta F_\alpha}{\delta\psi_\gamma}L_{\beta\gamma}\bar{C}_\beta-\bar{C}_\alpha\frac{\delta F_\alpha}{\delta\psi_\gamma}L_{\beta\gamma}\tilde{\psi}_\beta+\bar{C}_\alpha\frac{\delta^2 G_\alpha}{\delta\psi_\delta\,\delta\psi_\gamma}L_{\beta\delta}\bar{C}_\beta C_\gamma \tag{34}$$

This expansion may be reduced down by applying the operator

$$\frac{\delta}{\delta\bar{K}}\frac{\delta}{\delta\bar{K}}\frac{\delta F_\alpha}{\delta\psi_\delta}L_{\beta\delta}$$

to Eq. (32). The result is (again omitting integrations)

$$\left\langle\frac{1}{\varepsilon'}\delta S\right\rangle_{\mathscr{I}}=\left\langle\bar{K}_\gamma\frac{\delta F_\alpha}{\delta\psi_\delta}L_{\beta\delta}\bar{C}_\alpha\bar{C}_\beta\tilde{\psi}_\gamma\right\rangle_{\mathscr{I}}-\left\langle J_\gamma\frac{\delta F_\alpha}{\delta\psi_\delta}L_{\beta\delta}\bar{C}_\alpha\bar{C}_\beta C_\gamma\right\rangle_{\mathscr{I}} \tag{35}$$

Because $\langle\delta S\rangle_{\mathscr{I}}$ vanishes as $\mathscr{I}\to 0$, it could be said that the symmetry remains intact. However, $\langle\delta S\rangle_{\mathscr{I}}$ is nonvanishing, and so for all intents and purposes (with regard to calculating amendments to WTIs), SUSY is broken. In any case, through the usual manipulations the following WTI may be found:

$$\begin{aligned}&\langle\psi_2\dot{\psi}_1\rangle+\langle\psi_1 L_{\beta 2}\tilde{\psi}_\beta\rangle-\langle\psi_2 L_{\beta 1}\tilde{\psi}\rangle\\&\quad=-\int dx\,dy\left\langle C_1\frac{\delta N_\alpha(x)}{\delta\psi_\delta(y)}L_{\beta\delta}\bar{C}_\alpha(x)\,\bar{C}_\beta(y)\,C_2\right\rangle\end{aligned} \tag{36}$$

which reduces to the usual FDT as $N_\alpha\to 0$.

In the special case where $N_\alpha=\tau_{\alpha\beta}\psi_\beta$ the WTI becomes

$$\frac{1}{\Gamma}\langle\psi_2\dot{\psi}_1\rangle+\langle\psi_1\tilde{\psi}_2\rangle-\langle\psi_2\tilde{\psi}_1\rangle=(\tau_{\alpha\beta}-\tau_{\beta\alpha})\int dx\,\langle C_1\bar{C}_\beta(x)\,\tilde{\psi}_\alpha(x)\psi_2\rangle \tag{37}$$

which contains the usual FDT plus a new contribution on the RHS. This choice of F corresponds to a model[6] used to study freezing transitions in nonequilibrium systems; in that scenario $\tau_{\alpha\beta}$ is a Gaussian zero-mean matrix. Also of note is that only the antisymmetric parts of $\tau_{\alpha\beta}$ contribute a new term to the usual FDT, for the symmetric parts could be written as part of the Hamiltonian, and therefore not lead to a change in the FDT.

The relation of our equation to dissipation in the system seems to be obscured, similarly to the previous example. Consider the case where the RHS is decoupled, as a lowest-order approximation. After using $\langle C_1 C_2 \rangle = -\langle \psi_1 \tilde{\psi}_2 \rangle$ and Fourier transforming, it becomes

$$-i\omega C_{12}(\omega) + \chi_{12}(\omega) - \chi_{21}(-\omega) = \frac{(\tau_{\beta\alpha} - \tau_{\alpha\beta})}{\Gamma} \chi_{1\beta}(\omega)\, \chi_{2\alpha}(\omega) \qquad (38)$$

which now involves the real part of the susceptibility $\chi_{12}(\omega) \equiv \Gamma\langle \psi_1(\omega)\, \tilde{\psi}_2(-\omega)\rangle$, and the correlation function $C_{12} \equiv \langle \psi_1(\omega)\, \psi_2(-\omega)\rangle$. Note that since DB does not hold here it is not possible to write $\chi^{12}(\omega) = \chi^{21}(\omega)$ and simplify the expression. For now, it seems the most sensible interpretation of this case coincides with the previous example. That is, the term which makes the model nonequilibrium does not seem to participate in the same way as the other interaction terms; it remains as a source term, able top influence correlations at later times.

As a final example I consider a model that has been used to describe fast-ionic conductors, and is known as a driven-diffusive model.[7] Here a conserved density is driven to flow in a direction parallel to an external field E. Using equilibrium noise correlations, the model reads

$$\partial_t \psi = -F + v \qquad (39)$$

$$-F = D\Delta \frac{\delta H}{\delta \psi} - E \cdot \nabla \psi + v \qquad (40)$$

$$\langle v_x v_{x'} \rangle = -2D\, \Delta \delta(x - x') \qquad (41)$$

The action for this model is

$$S = \int dx \left\{ \tilde{\psi}(\dot{\psi} + F) + \tilde{\psi} D\, \Delta \tilde{\psi} - \bar{C} \int dy \left[\partial_t \delta(x-y) + \frac{\delta F}{\delta \psi_y} \right] C_y \right\} \qquad (42)$$

The field variations that correspond to SUSY (when $E = 0$) are

$$\begin{aligned} \delta\psi &= -\varepsilon C + \varepsilon' D\, \Delta \bar{C}, \qquad & \delta C &= -\varepsilon'(\dot{\psi} + D\, \Delta \tilde{\psi} \\ \delta\tilde{\psi} &= -\varepsilon' \dot{\bar{C}}, \qquad & \delta \bar{C} &= -\varepsilon \tilde{\psi} \end{aligned} \qquad (43)$$

Similar to as before, these lead to a WTI

$$\frac{1}{D} \langle \psi_2 \dot{\psi}_1 \rangle + \langle \psi_2 \Delta \tilde{\psi}_1) + \langle \psi_1 \Delta \tilde{\psi}_2 \rangle \qquad (44)$$

$$= \int dx\, \{ \langle C_1 \psi_2 \tilde{\psi} E \cdot \nabla\, \Delta \bar{C} \rangle - \langle C_1 \psi_2 \bar{C} E \cdot \nabla\, \Delta \tilde{\psi} \rangle \} \qquad (45)$$

[7] The particular form used for this example follows ref. 28.

which is here left in a symmetrized form. Aside from an interpretation as a residual source term, it is not clear at this point what other physical significance may be attributed to the new term.

Finally, I mention that the approach by which these WTIs were obtained applies equally well to the class of mode-coupling models (see Ref. 18 for a list of such models). They include, for example, models of a pure fluid at its gas–liquid critical point, as well as models for the critical behavior of antiferromagnets and superfluid helium. The standard way of writing mode-coupled equations is

$$\begin{aligned} \partial_t \phi_\alpha &= v_\alpha(\phi) - L_{\alpha\beta} \frac{\delta H}{\delta \phi_\beta} + \nu_\alpha \\ \langle \nu_\alpha(x)\, \nu_\beta(x') \rangle &= 2L_{\alpha\beta}\delta(x - x') \end{aligned} \tag{46}$$

where the $v_\alpha(\phi)$ are mode-coupling terms that are chosen under certain constraints.[29] Just as done earlier, the problem may be cast as a field theory and an action written down. Because the above equations are of exactly the same form as Eqs. (27)–(29), except with $N_\alpha(\phi)$ replaced with $v_\alpha(\phi)$, it is seen that there indeed is a set of WTIs when mode-coupling terms are present; they are given by Eq. (36) with v_α substituted for N_α.

In the special case where $v_\alpha = -M_{\alpha\beta}\, \delta H/\delta\phi_\beta$, with $M_{\alpha\beta}$ a constant (antisymmetric) matrix, the field variations (with $R_{\alpha\beta} = M_{\alpha\beta} + L_{\alpha\beta}$)

$$\begin{aligned} \delta\phi_\alpha &= \varepsilon C_\alpha + \varepsilon' R_{\beta\alpha} \bar{C}_\beta, \qquad & \delta C_\alpha &= \varepsilon'(\dot{\phi}_\alpha - R_{\beta\alpha}\tilde{\phi}_\beta) \\ \delta\tilde{\phi}_\alpha &= \varepsilon' \dot{\bar{C}}_\alpha, & \delta\bar{C}_\alpha &= \varepsilon\tilde{\phi}_\alpha \end{aligned} \tag{47}$$

leave $\delta S = 0$ [proceeding in the same way with $R_{\beta\alpha} = R_{\beta\alpha}(\phi)$ introduces new complications]. Following the same procedure as before, it is easy to show that

$$\langle \phi_2 \dot{\phi}_1 \rangle + \langle \phi_1 R_{\beta 2} \tilde{\phi}_\beta \rangle - \langle \phi_2 R_{\beta 1} \tilde{\phi}_\beta \rangle = 0 \tag{48}$$

This is of the same form[4] as the FDT for the more general case, where $R_{\alpha\beta}$ is a field-dependent matrix. Perhaps a lesson to be learned is that the choice of field variations must be made carefully, depending on which WTIs are sought.

4. CONCLUDING REMARKS

It is well known to practitioners in (and out) of nonequilibrium statistical mechanics that the field seriously lacks the theoretical scaffolding that has been so useful in corresponding equilibrium theories. What I have done here is derive an analog of an important result in equilibrium theories for these nontrivial nonequilibrium models. The technique may be readily applied to other Langevin models as well. It takes advantage of an approximate SUSY, and leads to nonperturbative relations among correlation functions. The new expressions I have found, which seem tol generalize the FDT in a sense, are of potential value in analyzing the models. The general interpretation made of the additional terms is that they are, loosely speaking, a leftover of detailed balance. They end up acting as source terms which contribute at the same level as thermal fluctuations. However, in distinction to noise, their effect must (in the present formulation) be integrated over the history of the system, for all times previous to those appearing explicitly in the WTI.

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